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Remarks on the quantum dilogarithm

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Abstract. A quantum analogue of the dilogarithm function has been introduced recently by Faddeev and Kashaev in such a way that a certain identity in the Weyl algebra $\overline{W}_q$ plays the role of the five-term dilogarithm identity. We study this identity in the limit when q approaches a root of unity and show that it then reduces to the 'restricted star–triangle relation' which has been used previously by Bazhanov and Baxter as a local integrability condition of a class of three-dimensional solvable lattice models.

1. Introduction

The dilogarithm function has an outstanding record of appearances in various branches of physics and mathematics. In mathematics, dilogarithms arise in number theory (the study of the asymptotic behaviour of partitions, e.g. [1, 2]; the values of Dedekind ζ-functions at $s = 2$ [3]); algebraic K-theory (the Bloch group [4]); the geometry of hyperbolic three-manifolds [5, 6]; the representation theory of Virasoro and Kac–Moody algebras [7] and conformal field theory [8, 9].

In physics, the dilogarithm appears, for instance, in the computation of the low-temperature asymptotics of the entropy [10, 11], and in calculations of the bulk free energy of certain two- and three-dimensional solvable lattice models [12–15]. Recently, the dilogarithm identities (through the thermodynamic Bethe ansatz) have appeared in the investigation of the ultraviolet limit and of the critical behaviour of integrable two-dimensional quantum field theories and lattice models [16–19] (see also [20] for the K-theoretical interpretation of these identities). Many properties of the dilogarithms mentioned above can be found in the survey [21].

Recall the definition and some basic properties of the dilogarithm. The Euler dilogarithm $\mathrm{Li}_2(x)$ is defined for $|x| < 1$ by

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{\log(1-t)}{t}\,\mathrm{d}t \tag{1.1}$$

and it can be analytically continued to the complex plane with the branch cut from 1 to ∞ along the real axis using the integral in (1.1). The related Rogers dilogarithm function $L(x)$ is defined as

$$L(x) = -\tfrac{1}{2}\int_0^x \left[\frac{\log(1-t)}{t} + \frac{\log t}{1-t}\right]\mathrm{d}t = \mathrm{Li}_2(x) + \tfrac{1}{2}\log x\,\log(1-x)\,. \tag{1.2}$$

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The most fundamental property of the dilogarithm function is the 'five-term' identity which in terms of $L(x)$ reads as

$$L(x)+L(y)=L\left(\frac{x(1-y)}{1-xy}\right)+L\left(\frac{y(1-x)}{1-xy}\right)+L(xy)\,. \tag{1.3}$$

Note that this relation, considered as an equation for the function $L(x) \in C^3[0,1]$, determines $L(x)$ up to a constant multiplier. When x and y are real and $0<x,y<1$ all the functions L in (1.3) are real and uniquely defined in the obvious sense by the integral in (1.2). By analytic continuation the relation (1.3), with a proper choice of the branches of L, remains valid for arbitrary complex x and y.

The five-term identity can be written in a number of different equivalent forms by using other more elementary relations for $L(x)$:

$$L(x)=-L(1-x)+\pi^2/6 \qquad L(x)=-L(1/x)+\pi^2/3 \tag{1.4}$$

$$L(x)=-L\left(\frac{-x}{1-x}\right) \tag{1.5}$$

where the last relation is a consequence of the first two. In particular, applying (1.5) for each term in (1.3) and then substituting $x/(x-1)$ and $y/(y-1)$ for x and y one obtains

$$L(x)+L(y)=L\left(\frac{y}{1-x}\right)+L\left(\frac{-xy}{1-x-y}\right)+L\left(\frac{x}{1-y}\right)\,. \tag{1.6}$$

Now let us describe the function $\mathrm{Li}_{2,q}(x)$ which will be considered as the quantum dilogarithm [22]. Consider the product

$$(x;q)_\infty=\prod_{k=0}^{\infty}(1-q^kx) \qquad |q|<1 \qquad x\in\mathbb{C}\,. \tag{1.7}$$

It is well known (see e.g. [23]) that

$$(x;q)_\infty=\sum_{n\geqslant 0}\frac{q^{n(n-1)/2}(-x)^n}{(q;q)_n} \qquad \frac{1}{(x;q)_\infty}=\sum_{n\geqslant 0}\frac{x^n}{(q;q)_n} \tag{1.8}$$

where

$$(x;q)_n=\prod_{k=0}^{n-1}(1-q^kx)\,. \tag{1.9}$$

Below we will use these functions as just formal power series over x.

Define the q-integral and the q-derivative in a standard way

$$\int_0^x f(t)\,\mathrm{d}_qt=x(1-q)\sum_{n=0}^{\infty}f(q^nx)q^n \qquad D_qf(x)=\frac{f(x)-f(qx)}{x(1-q)}\,. \tag{1.10}$$

We introduce the function $\mathrm{Li}_{2,q}(x)$ such that

$$\mathrm{Li}_{2,q}(x)=-\int_0^x\frac{\log(1-t)}{t}\,\mathrm{d}_qt \qquad D_q\mathrm{Li}_{2,q}(x)=-\frac{\log(1-t)}{t} \tag{1.11}$$

$$\lim_{q\to 1}\mathrm{Li}_{2,q}(x)=\mathrm{Li}_2(x)\,. \tag{1.12}$$

Then we can write $(x;q)_\infty$ as

$$(x;,q)_\infty=\exp(\mathrm{Li}_{2,q}(x)/(q-1))\,. \tag{1.13}$$

Now consider the Weyl algebra W over $\mathbb{C}[q, q^{-1}]$ generated by invertible elements u and v with the relation

$$uv = qvu \,. \tag{1.14}$$

Denote by $\overline{W}$ the following completion of this Weyl algebra. It is a vector space spanned by formal power series in u and v ordered in the following way:

$$a = \sum_{k,l\geqslant 0} a_{k,l}(q)u^k v^l \,. \tag{1.15}$$

The relation (1.14) determines multiplication in $\overline{W}$

$$ab = \sum_{m,n\geqslant 0} (ab)_{m,n}(q)u^n v^m \tag{1.16}$$

where

$$(ab)_{m,n} = \sum_{k,l\geqslant 0} a_{k,l}(q)b_{n-k,m-l}(q)q^{l(k-n)} \,. \tag{1.17}$$

Consider the following three remarkable relations between the elements of $\overline{W}$:

$$(u;q)_\infty(v;q)_\infty = (u+v;q)_\infty \tag{1.18}$$

$$(v;q)_\infty(u;q)_\infty = (u+v-vu;q)_\infty \tag{1.19}$$

$$(v;q)_\infty(u;q)_\infty = (u;q)_\infty(-vu;q)_\infty(v;q)_\infty \tag{1.20}$$

where $(a,q)_\infty$ with $a \in \overline{W}$ is defined by the power series (1.8). The first relation (1.18) is the well known property of the q-exponentials. The second relation (1.19) has been found recently in [24], while the third one is a simple consequence of the first two.

Faddeev and Kashaev [22] have recently given an interesting interpretation to this last relation (1.20), suggesting that it should be regarded as a quantum analogue the 'five-term' identity (1.3). Their main observation is as follows. Let us specify q to be a complex number, $|q| < 1$, and denote the corresponding specialization of the algebra $\overline{W}$ as $\overline{W}_q$. If one considers $\overline{W}_q$ as the result of the quantization of $C^\infty(\mathbb{R}^2)$ with standard Poisson structure on it, then the Wick symbol of the left- and right-hand sides of (1.20) can be computed explicitly in the limit $q \to 1$. The identity (1.20) then reduces to the five-term identity (1.3).

In this paper we study the relation (1.20) in the limit when q approaches a root of 1. We show that in this limit it provides a certain relation which had appeared before in the paper [15] as a local integrability condition of a class of three-dimensional solvable lattice models [25].

In section 2 we discuss the limit $q \to 1$. Section 3 contains the analysis of the limit $q \to \zeta$, where ζ is a primitive root of unity. In section 4 we identify the 'restricted star–triangle relation' from [15] with the relation which is the limit of (1.20) when $q \to \zeta$.

The main point of our calculations consists in a correct handling of singular elements in the centre of the algebra $\overline{W}_q$ in the limit $q \to 1$. A similar technique has recently been applied to other problems [26–28] related to the structure of quantum groups at roots of unity.

2. Quantum dilogarithm

In this section we consider the asymptotic form of the relations (1.18)–(1.20) in the limit $q \to 1$. For convenience let us set $q = \mathrm{e}^{-\tau}$.

At $q = 1$ the algebra $\overline{W}_q$ is commutative. However, the function $(x, q)_\infty$ has essential singularity at $\tau = 0$ (see equation (2.9) below) and to get a sensible limit of (1.18)–(1.20) one has to use both the asymptotic expansion (2.9) and the asymptotic expansion of the multiplication law in $\overline{W}_q$ when $\tau \to 0$.

Denote by $a * b$ the multiplication in $\overline{W}_q$ and by ab the multiplication in $\overline{W}_1$. The algebra $\overline{W}_q$ is a deformation of $\overline{W}_1$ (remember that $\overline{W}_q = \overline{W}_1$ as a vector space) and when $\tau \to 0$

$$a * b = ab + \tau m_1(a, b) + \tau^2 m_2(a, b) + \mathrm{O}(\tau^3)\,. \tag{2.1}$$

Functions $m_1(a, b)$, $m_2(a, b), \ldots$ can be explicitly computed from (1.16). The function $\{a, b\} = m_1(a, b) - m_1(b, a)$ determines the Poisson algebra structure on $\overline{W}_1$:

$$\{a, b\} = -\{b, a\} \tag{2.2}$$

$$\{ab, c\} = a\{b, c\} + b\{a, c\} \tag{2.3}$$

$$\{a, \{b, c\}\} + \{c, \{a, b\}\} + \{b, \{c, a\}\} = 0\,. \tag{2.4}$$

This Poisson bracket $\{\ ,\ \}$ is determined uniquely by its value on generators u and v:

$$\{u, v\} = -uv\,. \tag{2.5}$$

Let $H(a, b)$ be the Campbell–Hausdorff function defined as

$$\exp(-a/\tau) * \exp(-b/\tau) = F(a, b)\exp(-H(a, b)/\tau)(1 + \mathrm{O}(\tau)) \tag{2.6}$$

when $\tau \to 0$. Here $F(a, b)$ is some function which can be expressed merely in terms of $\{\ ,\ \}$ and $m_2(a, b)$.

Notice also that the adjoint action of the element $\exp(-a/\tau)$ in $\overline{W}_q$ becomes a Poisson action of $\exp(-a)$ as $q \to 1$,

$$\exp(-a/\tau) * b * \exp(a/\tau) = \exp(-a) \circ b\,(1 + \mathrm{O}(\tau)) \tag{2.7}$$

where

$$\exp(-a) \circ b = \sum_{n \geqslant 0} \frac{(-1)^n}{n!} \underbrace{\{a\{\cdots\{}_{n} a, b\} \cdots\}\,. \tag{2.8}$$

We are now ready to consider the relation (1.20) in the limit $q \to 1$. Applying the Euler–Maclaurin formula to the logarithm of $(x; q)_\infty$, one easily obtains the following asymptotic expansion:

$$(x; \mathrm{e}^{-\tau})_\infty = R(x)(1 + \mathrm{O}(\tau)) \qquad \tau \to 0 \tag{2.9}$$

$$R(x) = R(x, \tau) = (1 - x)^{1/2} \exp(-\mathrm{Li}_2(x)/\tau) \tag{2.10}$$

where all functions of x are understood as formal power series over x. Substituting this asymptotics into (1.20) one gets

$$R(v) * R(u) = R(u) * R(-vu) * R(v)(1 + \mathrm{O}(\tau))\,. \tag{2.11}$$

We want to compare the first two terms of the asymptotic expansions in τ of the logarithms of both sides of this relation. First, let us formulate some general statement. Suppose we have elements R_i in W_q which satisfy the relation

$$R_1 * R_2 = R_3 * R_4 * R_5 \tag{2.12}$$

in $\overline{W}_q$ with the asymptotics

$$R_i = M_i \mathrm{e}^{-L_i/\tau}(1 + \mathrm{O}(\tau)) \tag{2.13}$$

as $\tau \to 0$. Then for L_i and M_i we have

Proposition 2.1. The elements L_i and M_i satisfy the following relations:

$$H(L_1, L_2) = H(L_3, H(L_4, L_5)) = H(H(L_3, L_4), L_5) \tag{2.14}$$

$$M_1(\mathrm{e}^{-L_1} \circ M_2)F(L_1, L_2) = M_3(\mathrm{e}^{-L_3} \circ M_4)F(L_3, L_4)(\mathrm{e}^{H(L_3,L_4)} \circ M_5)F(H(L_3, L_4), L_5) \tag{2.15}$$

$$M_1(\mathrm{e}^{-L_1} \circ M_2)F(L_1, L_2) = M_3(\mathrm{e}^{-L_3} \circ (M_4(\mathrm{e}^{-L_4} \circ M_5)F(L_4, L_5)))F(L_3, H(L_4, L_5)))\,. \tag{2.16}$$

The proof is an easy corollary of (2.6) and (2.7). Conversely, it is quite clear that the relations (2.14)–(2.16) for element L_i and M_i imply the following relation:

$$R_1 * R_2 = R_3 * R_4 * R_5(1 + \mathrm{O}(\tau)) \tag{2.17}$$

for any elements R_i with asymtotics (2.13).

Now applying proposition 2.1 to (2.11) we obtain the following identity for the dilogarithm:

$$\begin{aligned} H(\mathrm{Li}_2(v), \mathrm{Li}_2(u)) &= H(\mathrm{Li}_2(u), H(\mathrm{Li}_2(-vu), \mathrm{Li}_2(v))) \\ &= H(H(\mathrm{Li}_2(u), \mathrm{Li}_2(-vu)), \mathrm{Li}_2(v))) \end{aligned} \tag{2.18}$$

and corresponding counterparts of the identities (2.15) and (2.16).

Obviously, the identity (2.18) can be interpreted as the equality of the singular in τ parts of the logarithms of the Wick symbols of left- and right-hand sides of (1.20). Remarkably, as was shown in [22], this identity is equivalent to the the five-term identity (1.3).

3. Quantum dilogarithm at roots of unity

Throughout this section we assume N to be an integer, $N \geqslant 2$, and

$$q = \mathrm{e}^{-\tau/N^2}\zeta \qquad \zeta^N = 1 \tag{3.1}$$

where ζ is a primitive root of unity of degree N, i.e. such that $\zeta^k \neq 1,\ 1 \leqslant k \leqslant N-1$.

Consider the algebra $\overline{W}_\zeta$. The following properties of $\overline{W}_\zeta$ are well known:

- The algebra $\overline{W}_\zeta$ has a centre $Z(\overline{W}_\zeta)$ generated by u^N and v^N.
- Let α and β be non-zero complex numbers and $I_{\alpha,\beta}$ be the ideal in $\overline{W}_\zeta$ generated by $(u^N - \alpha^N)$ and $(v^N - \beta^N)$, then for all $\alpha, \beta \in \mathbb{C}^*$

  $$\overline{W}_\zeta / I_{\alpha,\beta} \cong H_\zeta \tag{3.2}$$

  where H_ζ is the finite-dimensional algebra generated by the elements U, V obeying the relations

  $$UV = \zeta VU \qquad U^N = V^N = 1\,. \tag{3.3}$$

Let $a * b$ be the multiplication in $\overline{W}_q$ and ab the multiplication in $\overline{W}_1$:

$$a * b = a \cdot b + \tau m_1(a, b) + \tau^2 m_2(a, b) + \mathrm{O}(\tau^3)\,. \tag{3.4}$$

When at least one of the elements a or b belongs to the centre of $\overline{W}_\zeta$ define the bracket

$$\{a, b\} = m_1(a, b) - m_1(b, a)\,. \tag{3.5}$$

This bracket determines of the Poisson algebra structure on the centre $Z(\overline{W}_\zeta)$, together with its Poisson action on $\overline{W}_\zeta$:

$$\{a, bc\} = \{a, b\}c + b\{a, c\} \tag{3.6}$$

for $a \in Z(\overline{W}_\zeta)$ and $b, c \in \overline{W}_\zeta$.

Computing the bracket $\{a, b\}$ on generators from (3.4), one obtains

$$\{u, v^N\} = -uv^N/N \qquad \{u^N, v\} = -u^N v/N \qquad \{u^N, v^N\} = -u^N v^N . \tag{3.7}$$

Let $a \in Z(\overline{W}_\zeta)$ and $b \in \overline{W}_\zeta$ denote

$$\exp(a) \circ b = \sum_{n\geqslant 0} \frac{1}{n!} \underbrace{\{a\{\cdots\{a, b\}\cdots\}}_{n} = \lim_{\tau\to 0} \exp(a/\tau) * b * \exp(-a/\tau) \tag{3.8}$$

where $*$ is the multiplication in $\overline{W}_q$, $q = \mathrm{e}^{-\tau/N^2}\zeta$. Using equation (3.7) one can easily establish:

Proposition 3.1.

$$\begin{aligned} \exp\bigl(-\mathrm{Li}_2(u^N)\bigr) \circ v &= v(1-u^N)^{-1/N} \\ \exp\bigl(-\mathrm{Li}_2(v^N)\bigr) \circ u &= u\bigl(1-v^N\bigr)^{1/N} . \end{aligned} \tag{3.9}$$

Define the following function:

$$d(x) = d(x, \zeta, N) = (1-x^N)^{(N-1)/2N} \prod_{k=1}^{N-1} (1-\zeta^k x)^{-k/N} \tag{3.10}$$

where all the roots are understood as their series expansions in x at $x = 0$. This function is analytic for $|x| < 1$ and can be analytically continued to the cut complex plane with N branch cuts along the rays $\arg x = 2\pi k/N$, $k = 0, \ldots, N-1$, $|x| > 1$.

The following result holds:

Proposition 3.2. When $|x| < 1$, $q = \mathrm{e}^{-\tau/N^2}\zeta$, $\tau \to 0$, the function $(x; q)_\infty$ has the asymptotic form

$$(x; q)_\infty = (1-x^N)^{(1-N)/2N} R(x^N) d(x) (1 + \mathrm{O}(\tau)) \tag{3.11}$$

where R is defined by (2.10).

Proof. Write $(x; q)_\infty$ as

$$(x; q)_\infty = \prod_{k=0}^{N-1} (xq^k, q^N)_\infty \tag{3.12}$$

and use (2.9) for each of the N factors in the last product, subsequently expanding the result in a series in τ. Then use the formula

$$\mathrm{Li}_2(x^N) = N \sum_{k=0}^{N-1} \mathrm{Li}_2(\zeta^k x) \qquad |x| < 1 . \tag{3.13}$$

Now, let us consider the relation (1.20) when $q \to \zeta$. We will show that similarly to proposition 2.1 the leading terms in the asymptotics of (1.20) in this limit provide certain relations between functions in (3.11). The first one is a relation of the type (2.11) between elements of the centre $Z(\overline{W}_\zeta)$, while the second one (see equation (3.18) below) is a relation between the elements of $\overline{W}_\zeta$ expressed only through the function $d(x)$.

First, note that for any complex α

$$R(\mathrm{e}^{\alpha\tau} x) = (1-x)^\alpha R(x)(1 + \mathrm{O}(\tau)) . \tag{3.14}$$

Now substitute (3.11) into (1.20). Since for any primitive root of unity $\zeta^{N(N-1)/2} = (-1)^{(N-1)}$, we have

$$(-vu)^N = -\mathrm{e}^{-\tau(N-1)/2N} v^N u^N . \tag{3.15}$$

Using this relation for the second factor in the RHS of (1.20), rescaling the generators $(u, v) \to (\lambda u, \lambda v)$, with $\lambda = \exp((N-1)\tau/2N^2)$ and using (3.14) we obtain

$$R(v^N)d(v)R(u^N)d(u) = R(v^N)d(v)R(-v^Nu^N)d(-vu)R(u^N)d(u)(1+\mathrm{O}(\tau)). \tag{3.16}$$

Now, using (3.8) move all R's to the right. They then cancel due to (2.11). Hence for $d(x)$ one gets

$$d(v)\big(\mathrm{e}^{-\mathrm{Li}_2(v^N)} \circ d(u)\big) = d(u)\big(\mathrm{e}^{-\mathrm{Li}_2(u^N)} \circ d(-vu)\big)\big(\mathrm{e}^{-\mathrm{Li}_2(u^N)} \circ \mathrm{e}^{-\mathrm{Li}_2(-v^Nu^N)} \circ d(v)\big). \tag{3.17}$$

Using equation (3.9) and rescaling $u \to u/(1-v^N)^{1/N}$ we obtain:

Theorem 3.3. The following 'quantum five-term' identity holds:

$$d(v)d(u) = d\left(\frac{u}{(1-v^N)^{1/N}}\right) d\left(\frac{-vu}{(1-u^N-v^N)^{1/N}}\right) d\left(\frac{v}{(1-u^N)^{1/N}}\right) \tag{3.18}$$

where u, v, $uv = \zeta vu$, are the generators of $\overline{W}_\zeta$.

Remarkably, the Nth power of the arguments in (3.18) precisely match the arguments of the classical 'five-term' identity in the form (1.6).

Note that the relation (3.18) in an equivalent (but different) form has been given in [22]. It was noted therein that this relation is also equivalent to the 'restricted star–triangle relation' which has been used in [15] as a local integrability condition for the three-dimensional lattice models of [25]. We discuss this connection in the next section.

4. Restricted star–triangle relation

Following [15], let us formulate the 'restricted star–triangle relations'.

Let $\omega^{1/2}$ be a primitive root of -1 such that

$$(\omega^{1/2})^2 = \omega = \zeta^{-1} \qquad (\omega^{1/2})^N = -1. \tag{4.1}$$

Consider the algebraic curve in $\mathbb{C}^2$ (Fermat curve), defined as

$$x^N + y^N = 1. \tag{4.2}$$

For $n \in \mathbb{Z}$ define the meromorphic function $\overline{w}(x, y, n)$ on the curve (4.2) as

$$\overline{w}(x, y, n) = y^n \prod_{j=1}^{n} (1-\omega^j x)^{-1} \tag{4.3}$$

$$\overline{w}(x, y, N+n) = \overline{w}(x, y, n). \tag{4.4}$$

Consider the automorphism of the curve (4.2) given by the map:

$$(x, y) \to (\widetilde{x}, \widetilde{y}) \qquad \widetilde{x} = \omega^{-1}x^{-1} \qquad \widetilde{y} = \omega^{-1/2}y/x. \tag{4.5}$$

It is easy to check that

$$\overline{w}(x, y, n)\overline{w}(\widetilde{x}, \widetilde{y}, -n) = \Phi(n)^{-1} \tag{4.6}$$

where

$$\Phi(n) = (\omega^{1/2})^{n(N+n)} \qquad \Phi(n+N) = \Phi(-n) = \Phi(n). \tag{4.7}$$

Let $s_1, t_1, s_2, t_2, t_{12} \in \mathbb{C}$ satisfy the relations

$$s_1^N + t_1^N = s_2^N + t_2^N = s_1^N + s_2^N + t_{12}^N = 1. \tag{4.8}$$

Define five points (x_i, y_i), $i = 1, \ldots, 5$ on the curve (4.2) such that

$$\begin{aligned} &(x_1, y_1) = (s_1, t_1) && (x_2, y_2) = (t_2, \omega s_2) \\ &(x_3, y_3) = (t_2/\omega s_1, \omega^{-1/2} t_{12}/s_1) && (x_4, y_4) = (t_{12}/t_1, \omega s_2/t_1) \\ &(x_5, y_5) = (t_1 t_2/t_{12} . \end{aligned} \tag{4.9}$$

The following result was obtained in [15] (see equation (1.22) therein):

Theorem 4.1. The function $\overline{w}$ satisfies the following 'restricted star–triangle relation'

$$\frac{\overline{w}(x_2, a-c)}{\overline{w}(x_1, a-b)\overline{w}(x_3, b-c)} = \phi \sum_{d \in \mathbb{Z}_N} \overline{w}(x_4, a-d)\Phi(b-d)\overline{w}(x_5, d-c) \tag{4.10}$$

where $a, b, c, d \in \mathbb{Z}_N$, $\overline{w}(x_i, n) \equiv \overline{w}(x_i, y_i, n)$ with x_i, y_i defined above in (4.9) and ϕ is a normalization factor which does not depend on a, b and c.

The normalization factor ϕ can be easily calculated and we will do that later. Due to (4.4) and (4.7) the differences of the integers a, b, c and d in (3.18) can be interpreted *modulo* N. The relations among the arguments of $\overline{w}$ as they are presented here in (4.9) are equivalent to those in (1.21) of [15] (provided one corrects a minor misprint in the second formula in (1.21b) therein which should read $\Delta(v_2') = \omega v_1 \Delta(v_2/\omega v_1)/\Delta(v_2)$).

Consider the following matrix realization of the algebra $H_{\omega^{-1}}$ (see equation (3.3)):

$$U \to (U_{a,b}) \qquad V \to (V_{a,b}) \tag{4.11}$$

where

$$U_{a,b} = \delta_{a,b+1} \qquad V_{a,b} = \delta_{a,b}\omega^a \qquad a, b \in \mathbb{Z}_N \tag{4.12}$$

with

$$\delta_{a,b} = \begin{cases} 1 & a = b \pmod N \\ 0 & \text{otherwise} . \end{cases} \tag{4.13}$$

Then it follows from (3.2) that the generators u, v of $\overline{W}_{\omega^{-1}}$ can be written as

$$u = \alpha U \qquad v = \beta V \tag{4.14}$$

where $\alpha, \beta \in \mathbb{C}$. Consider the case

$$|\alpha^N| < |\Delta(\alpha)| < 1 \qquad |\beta^N| < |\Delta(\beta)| < 1 \tag{4.15}$$

$$|\alpha^N + \beta^N| < 1 \qquad |\alpha^N \beta^N| < |\Delta(\alpha, \beta)| \tag{4.16}$$

where

$$\Delta(\alpha) = \left(1 - \alpha^N\right)^{1/N} \qquad \Delta(\alpha, \beta) = \left(1 - \alpha^N - \beta^N\right)^{1/N} \tag{4.17}$$

and we fix the phases of the Nth roots requiring that $\Delta(0) = \Delta(0, 0) = 1$.

Consider the asymptotic expansion of (4.10) when $s_1, s_2 \to 0$ and express the parameters in (4.8) through α and β in (4.14) as follows:

$$s_1 = \alpha \qquad s_2 = \beta \qquad t_1 = \Delta(\alpha) \qquad t_2 = \Delta(\beta) \qquad t_{12} = \Delta(\alpha, \beta) . \tag{4.18}$$

Theorem 4.2. The asymptotic expansion of the restricted star–triangle relation (4.10) as $s_1, s_2 \to 0$ is equivalent to the quantum five-term identity (3.18) in the formal power series over u and v, provided the identification (4.18) holds.

Proof. Below, we will show that the required expansion of (4.10) is just the relation (3.18) written in a matrix form. Following [15], we introduce the functions

$$y = \Delta(x) = \left(1 - x^N\right)^{1/N} \tag{4.19}$$

$$w(x,0) = (\Delta(x))^{(1-N)/2} \prod_{j=1}^{N-1} (1-\omega^{-j}x)^{j/N} \tag{4.20}$$

$$w(x,n) = w(x,0)\overline{w}(x,\Delta(x),n) \tag{4.21}$$

as analytic continuations of formal power series over x into the whole complex plane with cuts from the points $x = \exp(2\pi ik/N)$, $k = 0,\ldots,N-1$ to infinity.

We will use the following properties [15] of the function $w(x,n)$:

Proposition 4.3. The function $w(x,n)$ satisfies the following relations when $|x| < 1$:

$$w(\omega^n x, 0) = w(x,n) \qquad \prod_{k\in\mathbb{Z}_N} w(x,k) = 1 \tag{4.22a}$$

$$\det\left(\sum_{k\in\mathbb{Z}_N} w(x,k)U^k\right) = (c_+\lambda(x))^N \tag{4.22b}$$

$$\det\left(\sum_{k\in\mathbb{Z}_N} (w(x,k))^{-1}U^k\right) = (c_-\lambda(x))^N \tag{4.22c}$$

$$\frac{1}{w(\Delta(x)/\omega,n)} = \frac{1}{c_+\lambda(x)} \sum_{k\in\mathbb{Z}_N} w(x,k)\omega^{nk} \tag{4.22d}$$

where

$$\lambda(x) = (x/\Delta(x))^{(N-1)/2} \tag{4.23}$$

$$c_+ = \prod_{j=1}^{N-1} (1-\omega^{-j})^{j/N} \qquad (c_+c_-)^N = \omega^{N(N-1)/2}. \tag{4.24}$$

The proof consists of straightforward calculations using the definitions (4.19)–(4.21).

It follows from (3.10) with $\zeta = \omega^{-1}$

$$d(x) = d(x,\zeta,N) = \frac{1}{\omega(x,0)}. \tag{4.25}$$

Using this and proposition 4.3 one can easily compute the following matrix elements:

Proposition 4.4. The following relations holds:

$$d(xV)_{a,b} = \delta_{a,b}\frac{1}{w(x,a)} \tag{4.26}$$

$$d(\Delta(x)U/\omega)_{a,b} = \frac{w(x,a-b)}{c_+\lambda(x)} \tag{4.27}$$

$$d(-\omega^{-3/2}\Delta(x)VU)_{a,b} = \frac{\Phi(a)\Phi^{-1}(b)w(x,a-b)}{c_+\lambda(x)}. \tag{4.28}$$

Moreover, from (4.22*b*) and (4.22*c*) it follows that

$$\det(d(xV)) = \det(d(xU)) = \det(d(xVU)) = 1. \tag{4.29}$$

Now set $a = 0$ in (4.10), use (4.6) for the $\overline{w}(x_3,-b)$ and replace all $\overline{w}(x,n)$ therein by $w(x,n)$. This merely changes the normalization factor. Remembering (4.18) and using proposition 4.4, we can rewrite (4.10) as a matrix equation

$$d(v)d(u) = \phi' d(u/\Delta(v))d(-vu/\Delta(u,v))d(v/\Delta(u)) \tag{4.30}$$

where $d(x)$ is defined by (3.10) with $\zeta = \omega^{-1}$. The normalization factor ϕ' is determined by taking determinants of both sides of (4.30) and using (4.29). This gives $\phi' = 1$ and (4.30) exactly reduces to (3.18). This concludes the proof of the theorem 4.2.

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